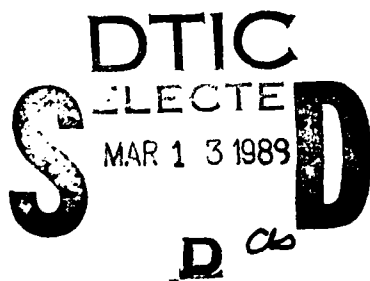AD-A205 914

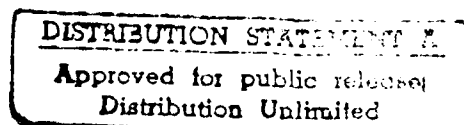# Projections of Vector Addition System Reachability Sets Are Semilinear

by

H. K. Büning, T. Lettmann, and E. W. Mayr

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# Projections of Vector Addition System Reachability Sets Are Semilinear

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## Abstract

The reachability sets of Vector Addition Systems of dimension six or more can be non-semilinear. This may be one reason why the inclusion problem (as well as the equality problem) for reachability sets of vector addition systems in general is undecidable, even though the reachability problem itself is known to be decidable. We show that any one-dimensional projection of the reachability set of an arbitrary vector addition system is semilinear, and hence "simple".

# 1 Introduction

Vector Addition Systems (VAS), and Petri nets, their equivalent graphical representation, are well-known models for the representation of aspects of parallel control, like liveness, or absence of deadlock. A central problem in their study is the description of the set of all states reachable in a given vector addition system, the so-called reachability set. In [7], it was shown that the membership problem for VAS reachability sets is decidable. For VAS's of dimension at most five, [4] establishes that their reachability sets are in fact semilinear, and gives an effective method for their construction. For dimensions above five, [4] also exhibits examples of VAS's with non-semilinear reachability sets.

The inherent complexity of VAS reachability sets is further underlined by the fact that the containment problem as well as the equality problem for VAS reachability sets are undecidable [1, 3], and by the exponential space lower bound known for the reachability problem [6, 8]. In particular, small "Petri net computers" can be built to simulate Turing machine computations using exponential space.

Often, we are interested in the values attainable for a certain component of a VAS. We will prove as the main result of this paper, that the projection of a VAS reachability set onto a single coordinate is always semilinear, and hence has a (relatively) simple structure. Of course, our result follows trivially if we restrict ourselves to VAS's with components of absolute value at most one. The problem is non-trivial, however, in the general case. A constructive proof of our result would, in fact, provide another proof of the decidability of the reachability problem since the latter is known to be recursively equivalent to the problem of deciding whether zero is reachable in a specified component [3].

A variant of Petri nets allows inhibitor arcs which make the firability of a transition dependent upon the condition that a place contain no tokens. It is well-known that the reachability problem for Petri nets with inhibitor arcs is undecidable [3]. We give an example of such nets for which certain components no longer have semilinear projections.

The remainder of this paper is organized as follows: in section 2, we give basic definitions and state the main result of the paper. Section 3 first states a number-theoretic Lemma which is essential for the subsequent proof of our result, and we conclude in section 4 outlining some possible extensions and open problems.

# 2 Basic Definitions and Main Result

We first introduce some basic concepts and notation. A *vector addition system* or *VAS* is a pair $(x, V)$. The vector $x \in \mathbf{N}^n$ is called the *initial* or *starting* vector, the integer $n$ is the *dimension* of the VAS, and $V \subset \mathbf{Z}^n$ is a finite set of *transition vectors*. The *reachability set* of a VAS $(x, V)$ is the smallest set $R(x, V)$ satisfying the following two properties: (i) $x \in R(x, V)$, and (ii) whenever $z \in R(x, V)$, $v \in V$, and $z + v \in \mathbf{N}^n$ then $z + v \in R(x, V)$; i.e., $R(x, V)$ is closed under addition of transition vectors as long as the sum has only nonnegative components. We also say that $z \in R(x, V)$ is *reachable* in $(x, V)$. A *transition sequence* $(v^{(i)})_{1 \leq i \leq t}$ of transition vectors $v^{(i)}$ is *applicable*

to some vector $y \in \mathbf{N}^n$ if $y + \sum_{j=1}^{i} v^{(j)} \in \mathbf{N}^n$ for all $i = 1, \ldots, t$. In this case, the vector $z = y + \sum_{j=1}^{t} v^{(j)}$ is called *reachable from* $y$ *in* $(x, V)$. To denote this, we also use $y \to_V z$. Thus, $R(x, V) = \{z : x \to_V z\}$.

For $x \in \mathbf{Z}^n$, we use $x_i$ to denote the $i$-th component of the vector $x$, and we let $R_i(x, V)$ stand for the projection $\{z_i : z \in R(x, V)\}$ of the reachability set $R(x, V)$ onto the $i$-th component. The linear ordering $\leq$ of the integers is extended, in the canonical way, to the partial ordering $\leq$ of $\mathbf{Z}^n$, i.e., for $x, y \in \mathbf{Z}^n$, we have $x \leq y$ iff $x_i \leq y_i$ for $1 \leq i \leq n$, and $x < y$ iff $x \leq y$ and $x$ and $y$ are different.

A *linear* subset $L$ of $\mathbf{N}^n$ is a set of the form

$$L = \{b + \sum_{i=1}^{t} n_i b^{(i)}; \ n_i \in \mathbf{N} \text{ for } i = 1, \ldots, t\}$$

for some vectors $b, b^{(1)}, \ldots, b^{(t)} \in \mathbf{N}^n$ (often, $b$ is called the *base* of $L$, and the $b^{(i)}$ its *periods*). A *semilinear* set is a finite union of linear sets.

Semilinear sets have many desirable properties. It turns out that they are exactly the sets describable within the first order logic of the natural numbers with addition, also known as Presburger Arithmetic [9]. Our main result states that projections of VAS reachability sets also fall into this class.

**Theorem 1** *Let $(x, V)$ be an arbitrary VAS of dimension $n$, and $i \in \{1, \ldots, n\}$. Then the projection $R_i(x, V)$ of the reachability set $R(x, V)$ is semilinear.*

We shall give a proof of this Theorem in the next section.

# 3 Proof of Main Theorem

We first state a couple of lemmata which are essential later on in the proof of our main result.

**Lemma 3.1** *Let $S$ be a subset of $\mathbf{N}$ such that there are $n_0, t > 0$ with the property that*

$$s \in S, \ s \geq n_0 \qquad \Rightarrow \qquad s + t \in S.$$

*Then the set $S$ is semilinear.*

**Proof:** Define, for $i = 0, \ldots, t - 1$,

$$S_i := \{s \in S; \ s \geq n_0, \ s \equiv i \bmod t\},$$

and set

$$S_f := \{s \in S; \ s < n_0\}.$$

Clearly, $S = S_f \cup \bigcup_{i=0}^{t-1} S_i$. Also, each $S_i$ is linear. For suppose that $S_i \neq \emptyset$, and let $m_i := \min\{s; \ s \in S_i\}$. Then $m_i \geq n_0$, and by the condition of the Lemma, $m_i + kt \in S_i$ for all $k \geq 0$. Since $s \in S_i$ in fact implies $s = m_i \bmod t$ we obtain

$$S_i = \{m_i + kt; \ k \geq 0\},$$

which is clearly linear.

Since $S_f$ is finite, and since semilinear sets are closed under (finite) union, $S$ is semilinear. $\square$

**Lemma 3.2** *Let $\alpha$ be an infinite sequence of vectors from $\mathbf{N}^n$. Then $\alpha$ contains an infinite, (with respect to $\leq$) nondecreasing subsequence.*

This Lemma sometimes is referred to as Dickson's Lemma [2]. We omit a proof which can easily be obtained by induction on the dimension $n$.

Next, we describe an algorithm to construct, for a given VAS $(x, V)$, its *reachability tree* $T(x, V)$. This reachability tree is a directed graph (not quite, but almost, a tree) whose edges are labelled with transitions $v \in V$, and whose nodes are labelled with so-called *pseudo-vectors* which we shall describe below. Our construction is similar to constructions given in [5, 7], with some technical adaptations.

*Pseudo-vectors* of an $n$-dimensional VAS are vectors in $(\mathbf{N} \cup \{\omega\})^n$, *i.e.*, $n$-dimensional vectors whose components are either nonnegative integers or the special element $\omega$. The intuitive meaning for $\omega$ is that it stands for a value that can become arbitrarily large. Formally, it satisfies the following rules:

$$\omega + m = m + \omega = \omega, \text{ for all } m \in \mathbf{Z}.$$

Applicability of (finite) transition sequences to pseudo-vectors, and reachability of one pseudo-vector from another are defined as for vectors.

```
algorithm T(x, V);
co given a VAS (x, V), this algorithm constructs its reachability tree T = T(x, V) oc

begin
    let T initially consist of node r, with label l(r) = x;
    mark r as active;
    while T contains some active node p (with label l(p)) do
        for every v ∈ V
            (i)   which is applicable to l(p), and
            (ii)  for which there is no edge labelled v out of p yet
        do
            add to T a new, active node q, with label l(q) = l(p) + v;
            add to T the arc (p, q), with label v;
            if there is a node q' with l(q') = l(q) on the simple path from r to p
            co a path is simple if it contains no cycle oc
```

**then**
 replace the arc $(p, q)$ by $(p, q')$ with label $v$, and delete $q$ from $T$
**elsf** there is a node $q''$ on the simple path from $r$ to $p$ such that
  (i)   $l(q)$ and $l(q'')$ have the same set of $\omega$-coordinates,
  (ii)   $l(q'') < l(q)$
**co** if there are several such $q''$ one is picked at random **oc**
**then**
 **co** we call $q$ an $\omega$-*node* **oc**
 add to $T$ a cycle, starting and ending in $q$, with the same edge sequence and edge labels as the path from $q''$ to $q$;
 label the nodes on this cycle, including $q$, the same as the corresponding nodes on the path from $q''$ to $q$, except that all coordinates $i$ where $(l(q''))_i < (l(q))_i$ become $\omega$;
 mark all nodes on the cycle as *active*
 **co** we call this cycle the $\omega$-*cycle* of $q$ **oc**
 **fi**
**od**;
 mark $p$ as *inactive*
**od**
**end** algorithm $T$.

As indicated in the specification of the algorithm, we shall refer to nodes where new $\omega$-coordinates are introduced as $\omega$-nodes, and we call the cycle attached to such a node when it is created its $\omega$-cycle.

**Theorem 2** *Algorithm $T$ terminates.*

**Proof:** Applying König's Lemma, it suffices to show that there cannot be an infinite simple path starting from the root $r$. Since, whenever node $q$ is reachable from node $p$ in $T$, label $l(q)$ contains at least the $\omega$-coordinates present in $l(p)$, and since the dimension $n$ of the VAS is finite, it even suffices to show that there can be no infinite simple path with all nodes having the same set of $\omega$-coordinates. The last claim follows immediately from Lemma 2, applied to the projection of the labels onto the non-$\omega$-coordinates, and the definition of the algorithm. □

**Lemma 3.3** *Suppose $z = l(p)$ is the label of some node $p$ in the reachability tree $T(x, V)$. Let $(v^{(i)})_{1 \leq i \leq t}$ be a transition sequence applicable to $z$. Then there is a unique path in $T(x, V)$ starting from $p$ whose edge label sequence is $(v^{(i)})_{1 \leq i \leq t}$.*

**Proof:** The proof is immediate from the statement of the algorithm, by induction on the length $t$ of the transition sequence. □

**Lemma 3.4** *Let $p$ be some $\omega$-node in the reachability tree $T(x, V)$, and let $(v^{(i)})_{1 \leq i \leq t}$ be the sequence of edge labels along its $\omega$-cycle, starting at $p$. Then, for $\delta = \sum_{j=1}^{t} v^{(j)}$, we have*

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**(i)** $\delta_i > 0$ *for all coordinates $i$ which become new $\omega$-coordinates at $p$;*

**(ii)** $\delta_i = 0$ *for all finite (i.e., non-$\omega$) coordinates of $l(p)$.*

*For coordinates $i$ which are already equal to $\omega$ at earlier nodes on the (simple) path from the root $r$ to $p$, $\delta_i$ can be arbitrary.*

**Proof:** Again, the claim of the Lemma follows immediately from the statement of the algorithm. $\Box$

As we have seen above, every vector sequence applicable to the initial vector $x$ corresponds to a unique path in $T(x, V)$, starting at the root. Conversely, it is clear that not every such path corresponds to an applicable vector sequence, due to the effect of $\omega$-coordinates. However, we are still able to infer the existence of certain applicable vector sequences from the structure of the reachability tree $T(x, V)$.

More precisely, let $p$ be some $\omega$-node in $T(x, V)$, let $p^{(1)}, \ldots, p^{(k)}$ be the other $\omega$-nodes on the simple path from $r$ to $p$, in order, and let $\gamma$ and $\gamma_{(i)}$ be the sequences of the edge labels around the $\omega$-cycles through $p$ and $p^{(i)}$, respectively. Also, let $\alpha_{(i)}$, for $i = 1, \ldots, k$, be the sequence of edge labels between $p^{(i-1)}$ ($r$ for $i = 1$) and $p^{(i)}$ along any path from the root to $p$, and let $\alpha$ be defined similarly between $p^{(k)}$ and $p$. For any vector sequence $\beta$ and $\beta'$, we shall use $\delta(\beta)$ to denote the sum of all the vectors in $\beta$, $\beta\beta'$ to denote the concatenation of $\beta$ and $\beta'$, and $\beta^k$ to denote the $k$-fold iteration of $\beta$.

**Lemma 3.5** *Let notation be as above. Then there are integers $n_1, \ldots, n_k \geq 0$, and $b_1, \ldots, b_k \geq 0$ such that, for every $t \geq 0$, the vector sequence $\beta^{(t)}$ defined as*

$$\beta^{(t)} = \alpha_{(1)}\gamma_{(1)}^{t \cdot n_1 + b_1}\alpha_{(2)} \cdots \alpha_{(k)}\gamma_{(k)}^{t \cdot n_k + b_k}\alpha\gamma^t$$

*is applicable to the initial vector $x$, and*

$$\left(\sum_{j=1}^{k} n_j \cdot \delta(\gamma_{(j)}) + \delta(\gamma)\right)_i > 0$$

*for all $\omega$-coordinates of $l(p)$.*

**Proof:** The proof is by induction on $k$.

$k = 0$: The claim follows since there are no $\omega$-coordinates in labels of nodes above $p$ in the tree, and because of Lemma 4.

$k > 0$: Suppose the Lemma is true for all values up to $k - 1$, and let $n_1', \ldots, n_{k-1}'$ and $b_1', \ldots, b_{k-1}'$ be numbers as claimed by the Lemma. Let

$$\beta'^{(t)} = \alpha_{(1)}\gamma_{(1)}^{t \cdot n_1' + b_1'}\alpha_{(2)} \cdots \alpha_{(k-1)}\gamma_{(k-1)}^{t \cdot n_{k-1}' + b_{k-1}'}\alpha_{(k)}\gamma_{(k)}^t.$$

Also, let $I'$ be the set of $\omega$-coordinates of $l(p^{(k)})$, and $I$ the set of coordinates which become new $\omega$-coordinates at $p$. Note that the $i$-th coordinate of $\sum_{j=1}^{k-1} n_j' \cdot \delta(\gamma_{(j)}) + \delta(\gamma_{(k)})$ is greater

than zero for all coordinates in $I'$, by the second part of the induction hypothesis, and equal to zero for all other coordinates, because of Lemma 4. Hence, there are minimal $t'$ and $n'$ such that (i) the vector sequence $\alpha\gamma$ is applicable to $x + \delta(\beta'^{(t')})$, and (ii) the $i$-th component of $n' \cdot (\sum_{j=1}^{k-1} n'_j \cdot \delta(\gamma_{(j)}) + \delta(\gamma_{(k)})) + \delta(\gamma)$ is positive for all $i \in I'$. Note that the components in $I$ of this last sum are automatically greater than zero. We claim that $b_i = t' \cdot n'_i + b'_i$ for $i = 1, \ldots, k-1$, $b_k = t'$, $n_i = n' \cdot n'_i$ for $i = 1, \ldots, k-1$, and $n_k = n'$ satisfy the Lemma. Since the second part of the Lemma is true by construction we only have to show that $\beta^{(t)}$ is applicable to the initial vector $x$ for all $t$. For $t \leq 1$ this is immediate from the construction, and for $t > 1$ we note that

$$x + \delta(\beta'^{(tn'+t')}\alpha\gamma^{t-1}) \geq x + \delta(\beta'^{((t-1)n'+t')}\alpha\gamma^{t-2}),$$

and since $\gamma$ is applicable to the latter, so it is to the former. But this means in fact that $\beta^{(t)}$ is applicable to $x$, concluding our proof. $\square$

**Lemma 3.6** *Let $\alpha$ be any vector sequence applicable to $x$ such that the path corresponding to $\alpha$ in $T(x, V)$ contains an $\omega$-node where the $i$-th coordinate becomes $\omega$. Then there is some $t > 0$ such that $R_i(x, V)$ contains $(x + \delta(\alpha))_i + kt$, for all $k \geq 0$.*

**Proof:** Decompose $\alpha$ into $\alpha'\alpha''$ where $\alpha'$ is the prefix of $\alpha$ of maximal length ending at the $\omega$-node where coordinate $i$ becomes an $\omega$-coordinate. Then the claim follows from Lemma 5 and Lemma 4. $\square$

We are now able to finish the proof of Theorem 1. We partition the vector sequences applicable to the initial vector $x$ into classes $C_p$ and $C_0$: the class $C_p$, with $p$ an $\omega$-node in $T(x, V)$ where the $i$-th coordinate becomes $\omega$, is to contain all applicable vector sequences whose corresponding path in $T(x, V)$ contains $p$, and $C_0$ all remaining vector sequences. Clearly, by the finiteness of the reachability tree, the set $S_0 = \{(x + \delta(\alpha))_i; \alpha \in C_0\}$ is finite. Also, by Lemma 1 and Lemma 6, each of the sets $S_p = \{(x + \delta(\alpha))_i; \alpha \in C_p\}$ is semilinear, and hence so is the union of the $S_p$ and $S_0$. Since this union is equal to $R_i(x, V)$ the proof of Theorem 1 is complete.

As an immediate consequence of Theorem 1 we also obtain that the set of all integers which are equal to the sum of the components of a vector in the reachability set is semilinear.

**Corollary** *Let $(x, V)$ be an arbitrary VAS of dimension $n$. Then the set*

$$R_\Sigma(x, V) := \{\sum_{i=1}^{n} (y)_i; y \in R(x, V)\}$$

*is semilinear.*

**Proof:** Modify the given vector addition system $(x, V)$ by introducing the sum of the components of a vector as an additional component. An application of Theorem 1 on the resulting vector addition system $(x', V')$ defined by

$$x' := (\sum_{i=1}^{n} (x)_i, x_1, \ldots, x_n)$$

and

$$V'' := \{(\sum_{i=1}^{n} (y)_i, y_1, \ldots, y_n) : y \in V\}$$

gives us the semilinearity of $R_1(x', V'') = R_\Sigma(x', V'')$. □

# 4    Conclusion

In the terminology of Petri nets Theorem 1 means that the set of possible numbers of tokens for a certain place of a Petri net starting with some initial marking is semilinear. (For definitions see [3, 5].)

Allowing inhibitor arcs leads to a variant of Petri nets with undecidable reachability problem [3]. In contrast to ordinary Petri nets there are examples of these nets with non-semilinear projections for certain components.

*Example:* Petri nets with inhibitor arcs can be used to simulate programs via simulation of register machine computations, and certain projections of the reachability set starting with a certain initial marking correspond to the set of all values of a variable taken during the run of the program.

In the following program the variable $d$ takes all values in the non-semilinear set

$$A := \{d \in \mathbf{N} : d \geq 2, \exists c \geq 1 : d = c + length(c)\},$$

where $length(c) = \lceil \log_2(c + 1) \rceil$ is the length of the binary representation of $c$:

```
d := 2;
c := 1;
while true do
   c := c + 1;
   if (∃r : c = 2^r) then
      d := d + 2
   else
      d := d + 1
   fi
od.
```

Obviously a certain projection of the reachability set of the simulating Petri net will be the non-semilinear set $A$.

The above net, however, will contain several inhibitor arcs. In case of only one inhibitor arc it is still open, whether there exists a net and a place in it with a non-semilinear set of markings. Finally, it should also be mentioned that the reachability problem for nets with only one inhibitor arc is still open.

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